A Finite-Difference Time-Domain Method Without the Courant Stability Conditions

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***Abstract*—In this paper, a finite-difference time-domain method that is free of the constraint of the Courant stability condition is presented for solving electromagnetic problems. In it, the alternating direction implicit (ADI) technique is applied in formulating the finite-difference time-domain (FDTD) algorithm. Although the resulting formulations are computationally more complicate than the conventional FDTD, the proposed FDTD is very appealing since the time step used in the simulation is no longer restricted by stability but by accuracy. As a result, computation speed can be improved. It is found that the number of iterations with the proposed FDTD can be at least three times less than that with the conventional FDTD with the same numerical accuracy.**

***Index Terms*— Alternating direct implicit technique (ADI), FDTD method, instability, unconditional stable.**

I. Introduction

The finite-difference time-domain (FDTD) method [1] has been proven to be an effective means that provides accurate predictions for varieties of electromagnetic interaction problems [2]. Nevertheless, the FDTD is very memory and CPU-time intensive and consequently is not suitable for large-scale problems. Such intensive memory and CPU time requirements come from two reasons: 1) the spatial increment steps must be small enough in comparison with the wavelength (usually 10–20 steps per wavelength) in order to make the numerical dispersion error negligible, and 2) the time step must be small enough to satisfy the following stability condition (the Courant condition):

$$u_{\max}\Delta t \leq \left[\frac{1}{\Delta x^2}+\frac{1}{\Delta y^2}+\frac{1}{\Delta z^2}\right]^{-1/2}. \tag{1}$$

Here $u_{\max}$ is the maximum wave phase velocity within the model.

Various time-domain techniques have been developed to improve the FDTD computation efficiency. One of them is the recently developed multiresolution time-domain (MRTD) method. By using orthonormal wavelet spatial expansions, the MRTD scheme [3] can reduce the spatial discretization to two steps per wavelength. However, the stability condition for MRTD becomes more stringent. The time to spatial step ratio becomes five times less than that with the conventional FDTD. Another technique is the so-called pseudospectral time-domain (PSTD) method [4]. By using the fast Fourier transform (FFT) to represent spatial derivatives, the PSTD method can also achieve a grid arrangement of two steps per wavelength.

In this paper, a FDTD method without the Courant stability condition is presented. It is based on the Yee's grid but with the implementation of the alternative direction implicit technique that has been widely used to solve parabolic partial differential equation [5]. As a result, the time step used in the simulation is no longer restricted by stability but by accuracy of the algorithm. The numerical results indicate that with the same accuracy, the proposed FDTD method uses at least three times fewer of iterations and is at least 1.55 times faster than the conventional FDTD.

II. The Proposed FDTD Scheme

In an isotropic medium, Maxwell's curl vector equations can be represented by a system of six scalar Cartesian equations. For example, let us consider

$$\frac{\partial E_x}{\partial t} = \frac{1}{\epsilon}\left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right). \tag{2}$$

The proposed FDTD method consists of the following discretization process.

1) At the $(n+1)$th time step, only $\partial H_z/\partial y$, the *first* term on the right-hand side, is replaced with an implicit difference approximation in terms of the unknown pivotal values at the $(n+1)$th time step, while $\partial H_y/\partial z$, the *second* term on the right-hand side, is replaced with an explicit finite difference approximation in terms of the known values at the previous nth time step.
2) At the $(n+2)$th time step, $\partial H_y/\partial z$ (the *second* term) is replaced by an implicit finite-difference approximation in terms of the unknown pivotal values at the $(n+2)$th time step while the $\partial H_y/\partial y$ (the *first* term) with an explicit one in terms of the known values at the previous $(n+1)$th time step.

In other words, with the well-known Yee's finite difference grid arrangement, (2) is computed in two steps:

$$\frac{{}_{n+1}E^{x}+_{i+1/2,j,k} - {}_{n}E^{x}_{i+1/2,j,k}}{\Delta t} = \frac{1}{\epsilon}\left[\frac{{}_{n+1}H^{z}_{i+1/2,j+1/2,k} - {}_{n+1}H^{z}_{i+1/2,j-1/2,k}}{\Delta y} - \frac{{}_{n}H^{y}_{i+1/2,j,k+1/2} - {}_{n}H^{y}_{i+1/2,j,k-1/2}}{\Delta z}\right] \tag{3}$$

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is used to advance the solution from the nth to the $(n+1)$th time step, and

$$\frac{{}_{n+2}E^{x}_{i+1/2,j,k} - {}_{n+1}E^{x}_{i+1/2,j,k}}{\Delta t} = \frac{1}{\epsilon}\left[\frac{{}_{n+1}H^{z}_{i+1/2,j+1/2,k} - {}_{n+1}H^{z}_{i+1/2,j-1/2,k}}{\Delta y} - \frac{{}_{n+2}H^{y}_{i+1/2,j,k+1/2} - {}_{n+2}H^{y}_{i+1/2,j,k-1/2}}{\Delta z}\right] \tag{4}$$

for the advancement from the $(n+1)$th to the (n+2)th time step.

The notations ${}_{n}E^{\alpha}_{i,j,k}$, ${}_{n}H^{\alpha}_{i,j,k}$ with $\alpha = x, y, z$ are the field components with their positions in the Yee's grid being the same as the conventional FDTD.

Similar expressions can be derived for the other components, E_y, E_z, H_x, H_y and H_z at the $(n+1)$th and $(n+2)$th time-step, respectively. By substituting the expression for H_z at the $(n+1)$th time-step into (3), one can have

$$\begin{aligned} &-\left(\frac{\Delta t^2}{\mu\epsilon\Delta y^2}\right){}_{n+1}E^{x}_{i+1/2,j+1,k} + \left(1+\frac{2\Delta t^2}{\mu\epsilon\Delta y^2}\right){}_{n+1}E^{x}_{i+1/2,j,k} \\ &- \left(\frac{\Delta t^2}{\mu\epsilon\Delta y^2}\right){}_{n+1}E^{x}_{i+1/2,j-1,k} \\ &= {}_{n}E^{x}_{i+1/2,j,k} + \frac{\Delta t}{\epsilon\Delta y}\left({}_{n}H^{z}_{i+1/2,j+1/2,k} - {}_{n}H^{z}_{i+1/2,j-1/2,k}\right) \\ &\quad - \frac{\Delta t}{\epsilon\Delta z}\left({}_{n}H^{y}_{i+1/2,j,k+1/2} - {}_{n}H^{y}_{i+1/2,j,k-1/2}\right) \\ &\quad - \frac{\Delta t^2}{\mu\epsilon\Delta y\Delta x}\left({}_{n}E^{y}_{i+1,j+1/2,k} - {}_{n}E^{y}_{i,j+1/2,k} \right. \\ &\qquad \left. - {}_{n}E^{y}_{i+1,j-1/2,k} + {}_{n}E^{y}_{i,j-1/2,k}\right). \end{aligned} \tag{5}$$

The equations for the other five components can be derived in a similar way. Note that in the above equation, there is no half time-step difference between electric and magnetic field components.

The above recursive equation can be solved either implicitly or explicitly. In an actual computation, a recursive scheme can be used. For example, consider (5). Suppose that the leftmost values of E_x, say at $j=0$ and $j=1$ at the $(n+1)$th time step, are obtained. The rest E_x's can be calculated by applying (5) with a sequence of ascending j that allows us to find E_x at $j+1$ from E_x at j and $j-1$. In such a way, the computation efficiency can be improved.

Numerical Stability

It can be proved theoretically that the proposed scheme is inherently unconditionally stable or without the constraint of the Courant stability condition. Due to the limit of the space in this letter, the details of the theoretical proof are not shown here. However, an experiment was performed to numerically show the proposed scheme is stable as described in the following section.

III. Numerical Results

For the sake of simplicity and verifications, a rectangular cavity was computed with the proposed FDTD scheme. The cavity has the dimension of 9 mm $\times$ 6 mm $\times$ 15 mm. A uniform mesh with $\Delta l = 0.6$ mm was used, leading to a total number of 10 $\times$ 15 $\times$ 25 grid points.

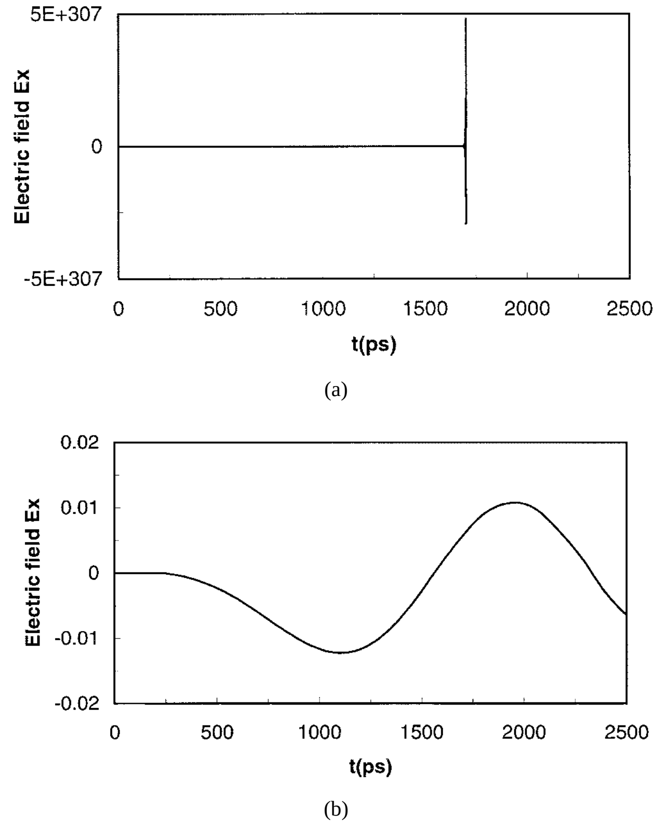

Fig. 1. Time-domain electric fields with the conventional FDTD and the proposed FDTD. (a) The conventional FDTD solutions that becomes unstable with $\Delta t_j = 1.2$ ps. (b) The proposed FDTD solution with $\Delta t_i = 120$ ps.

A. Numerical Verification of the Stability

To verify numerically that the proposed FDTD scheme is unconditional stable, simulations with the conventional FDTD and the proposed FDTD were run with a time step, Δt, exceeding the time step limit for the stable conventional FDTD algorithm that is $\Delta t_{FDTDMAX} = (\Delta l/c\sqrt{3}) = 1.155$ ps in this case. Fig. 1 shows the electric field recorded at the center of the cavity. $\Delta t_{FDTD} = 1.2$ ps was used with the conventional FDTD while $\Delta t_i = 120$ ps (that is 100 times of 1.2 ps) was used with the proposed FDTD scheme. As can be seen, the conventional FDTD quickly becomes unstable [see Fig. 1(a)], while the proposed FDTD still gives stable results [see Fig. 1(b)].

B. Accuracy Versus Time Step

Since the proposed FDTD is always stable, the selection of the time step is no longer restricted by stability but by modeling accuracy. As a result, it is meaningful to investigate how the time step will affect accuracy.

For the comparison purpose, both the conventional FDTD and the proposed FDTD were used to simulate the cavity again. This time, the time step $\Delta t_{FDTD} = 0.8$ ps was chosen with

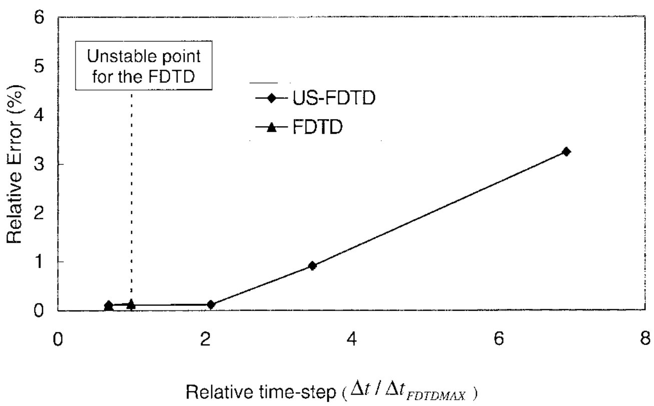

Fig. 2. Relative errors of the conventional FDTD and the proposed FDTD as the function of relative time step $\Delta t/\Delta t_{FDTDMAX}$. Dash line represents the unstable point of FDTD scheme.

the conventional FDTD while the variable time step Δt_i was chosen with the proposed FDTD to check on the accuracy.

Fig. 2 illustrates the relative errors computed for the dominant mode of the cavity using the conventional FDTD and the proposed FDTD with variable time steps. For clarity, relative time-step $\Delta t/\Delta t_{FDTDMAX}$ is used. As can be seen, at low $\Delta t/\Delta t_{FDTDMAX}$, the errors of both the conventional FDTD and the proposed FDTD are almost the same. However, after $\Delta t/\Delta t_{FDTDMAX} = 1.0$, the conventional FDTD solution becomes diverge (unstable) while the proposed FDTD continues to produce stable results with increasing errors.

C. Computation Memory and CPU Time Saving

Again, for the comparison purpose, both the conventional FDTD and proposed FDTD were used to simulate the cavity. This time, the time step $\Delta t_{FDTD} = 0.8$ ps was chosen with the conventional FDTD while the time step $\Delta t_i = 3 \times \Delta t_{FDTD} = 2.4$ ps was chosen with the proposed FDTD. The reason for such time step selections is that they will provide similar accuracy with the two methods. The two methods can then be compared in a fair manner. Twelve hundred iterations was run with the conventional FDTD and 400 iterations with the proposed FDTD method. As a result, the physical time periods simulated by the two methods are the same (since the time step with the conventional FDTD is one-third of that with the proposed FDTD). Table I shows the resonant frequencies obtained with the conventional FDTD and the proposed FDTD. As can be seen, the errors for both methods are at the same level.

TABLE I
COMPARISONS OF RESULTS WITH THE CONVENTIONAL FDTD AND THE PROPOSED FDTD

Analytic results (GHz)	Conventional FDTD scheme		Proposed FDTD scheme	
	Simulation results (GHz)	Relative error	Simulation results (GHz)	Relative error
19.427	19.451	0.12%	19.400	0.14%
26.022	25.972	0.19%	25.961	0.23%
31.652	31.455	0.62%	31.553	0.31%
34.776	34.613	0.47%	34.577	0.57%

On a Pentium 166-MHz PC, it took 58.97 s to finish with the conventional FDTD and 38.13 s with the proposed FDTD. It is then concluded that a saving of 1.55 times with the proposed FDTD was achieved in our case.

IV. CONCLUSION

A three-dimensional (3-D) FDTD method was presented for solving electromagnetic problems. In it, the Yee's grid is used but the alternative direction technique is applied in formulating the algorithm. As a result, the stability condition associated the FDTD method is removed. Time step used is then solely restricted by the accuracy of the numerical discrete models. Preliminary numerical results showed the validity of the method. Theoretical roof of the unconditional stability and theoretical investigations on accuracy including numerical dispersion of the FDTD method will be presented in our future publications.

It should be noted that very recently, the 2-D Courant-condition free FDTD was proposed in [6]. This letter extends the method to three dimensions with numerical validations.

REFERENCES

[1] K. S. Yee, "Numerical solution of initial boundary value problems involving Maxwell's equations in isotropic media," *IEEE Trans. Antennas Propagat.*, vol. AP-14, pp. 302–307, May 1966.

[2] A. Taflove, *Computational Electrodynamics: The Finite-Ddifference Time-Domain Method.* Norwood, MA: Artech House, 1996

[3] M. Krumpholz and L. P. B. Katehi, "MRTD: New time-domain schemes based on multiresolution analysis," *IEEE Trans. Microwave Theory Tech.*, vol. 44, pp. 555–571, Apr. 1996.

[4] Q. H. Liu, "The pseudospectral time-domain (PSTD) method: A new algorithm for solution of Maxwell's equations," in *Proc. IEEE Antennas and Propagation Society Int. Symp.*, vol. 1, pp. 122–125, 1997.

[5] D. W. Peaceman and H. H. Rachford, "The numerical solution of parabolic and elliptic differential equations," *J. Soc. Ind. Appl. Math.*, vol. 42, no. 3, pp. 28–41, 1955.

[6] T. Namiki and K. Ito, "A new FDTD algorithm free from the CFL condition restraint for a 2D-TE wave," *Dig. 1999 IEEE Antennas and Propagation Symp.*, July 11–16, 1999, Orlando, FL, pp. 192–195.